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STABILITY OF THIN SHALLOW SHELLS OF NEGATIVE GAUSSIAN CURVATURE

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The density of eigennumbers in stability problems of shells with positive Gaussian curvature is examined in [1-3]. An interpretation of the results obtained was proposed that permits relating the density of the initial section of the spectrum to the shell responsiveness to small perturbations during experiment, and also to imperfections in the geometric shape of the shell. Investigation of the spectrum in problems studied less, stability problems of shells of negative Gaussian curvature, is natural. Of greatest interest are shells of negative Gaussian curvature that are almost cylindrical.

The system of stability equations of shallow shells whose radii are almost constant has the form [4]

$$(\mathrm{Eh})^{-1}\nabla^2\nabla^2\varphi - \Delta_k^2 w = 0,\ D\nabla^2\nabla^2 w + \Delta_k^2\varphi = \sigma\nabla^2\nabla^2(\alpha_1 w_{,xx} + \alpha_2 w_{,yy}),$$
$$\sigma\alpha_1 = -T_1,\ \sigma\alpha_2 = -T_2,\ \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2},\ \Delta_k^2 = \frac{1}{R_2}\frac{\partial^2}{\partial x^2} + \frac{1}{R_1}\frac{\partial^2}{\partial^2 y^2},$$

where x, y are Cartesian coordinates; w(x, y), normal deflection; φ(x, y), stress function; T_1, T_2, forces in the shell middle surface; and $R_1 \approx$ const, $R_2 \approx$ const. The eigenfunctions of stability problems of hinge-supported panels have the form

$$\varphi(x,\ y) = \varphi_0 \sin k_m x \sin k_n y,$$
$$w(x,\ y) = w_0 \sin k_m x \sin k_n y,\quad k_n = n\pi/a,\ k_m = m\pi/b,$$
$$n,\ m = 1,\ 2,\ \ldots$$

The eigenfunctions for shells of revolution are also the following

$$\varphi(x,\ y) = \varphi_0 \sin k_m x \cos k_n y,$$
$$w(x,\ y) = w_0 \sin k_m x \cos k_n y, \tag{1}$$

$k_n = n/R$, $k_m = m\pi/l$, n = 0, 1, ..., m = 1, 2, ... For n = 0, we obtain the eigenfunctions of axisymmetric buckling from the relationships (1). The eigennumbers of the problem under consideration are found from the formula

$$\lambda_{mn} = \frac{(k_m^2 + k_n^2)^4 + \varkappa^4 (k_m^2 + \chi k_n^2)^2}{(k_m^2 + \vartheta k_n^2)(k_m^2 + k_n^2)^2}, \tag{2}$$

where $\lambda = -\sigma\alpha_1/D$, $\varkappa^4 = Eh/DR_2^2$, $\chi = R_2/R_1$, $\vartheta = \alpha_2/\alpha_1$.

Let us introduce a polar coordinate system

$$k_m = r\cos\theta,\ k_n = r\sin\theta\ (r \geqslant 0,\ 0 \leqslant 0 \leqslant \pi/2) \tag{3}$$

in the plane of the wave numbers k_m, k_n. After substituting these expressions for k_m, k_n into (2), we obtain a biquadratic equation in the polar radius r. After still another substitution $\xi = \sin^2\theta$ $(0 \leqslant \xi \leqslant 1)$, the formula for r takes the form $(\eta = \lambda/2\varkappa^2)$

$$r_{1,2}^2 = \varkappa^2\{\eta[1 - \xi(1-\vartheta)] \pm \sqrt{\eta^2[1-\xi(1-\vartheta)]^2 - [1-\xi(1-\chi)]^2}\}. \tag{4}$$

The relationship (4) determines the boundary of the domain Ω within which $\eta < \eta_0$. It is meaningful under the condition

$$\eta^2[1 - \xi(1-\vartheta)]^2 - [1 - \xi(1-\chi)]^2 \geqslant 0. \tag{5}$$

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The number of eigenvalues A less than a given η_0 is defined in [5] as the ratio between the area of the domain Ω within which $\eta < \eta_0$ and the area of a cell $\Delta k_m \Delta k_n$:

$$A(\eta) = \frac{1}{\Delta k_m \Delta k_n} \iint_\Omega dk_m dk_n. \tag{6}$$

The relationship (6) in the coordinate system (3) has the form

$$A(\eta) = \frac{ab\varkappa^2}{\pi^2} \int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} r dr d\theta.$$

After integration with respect to r and substitution of (4), we obtain

$$A(\eta) = \frac{ab\varkappa^2}{2\pi^2} \int_{S_1(\eta)}^{S_2(\eta)} \sqrt{\frac{\eta^2[1-\xi(1-\vartheta)]^2 - [1-\xi(1-\chi)]^2}{\xi(1-\xi)}} d\xi. \tag{7}$$

The limits of integration $S_1(\eta)$ and $S_2(\eta)$ are determined from condition (5). The expression for the eigenvalue density is obtained by differentiating (7) with respect to the parameter η:

$$N(\eta) = \frac{ab\varkappa^2}{2\pi^2} \int_{S_1(\eta)}^{S_2(\eta)} \frac{\eta[1-\xi(1-\vartheta)]^2 d\xi}{\sqrt{\xi(1-\xi)\{\eta^2[1-\xi(1-\vartheta)]^2 - [1-\xi(1-\chi)]^2\}}}. \tag{8}$$

Let us introduce the notation

$$\xi_1 = (1-\eta)/(1-\chi-\eta(1-\vartheta)),\ \xi_2 = (1+\eta)/(1-\chi+\eta(1-\vartheta)).$$

We can write (8) in the form

$$N(\eta) = \frac{ab\varkappa^2}{2\pi^2 B} I,\ I = \int_{S_1(\eta)}^{S_2(\eta)} \frac{\eta[1-\xi(1-\vartheta)]^2 d\xi}{\sqrt{a_0\xi(1-\xi)(\xi-\xi_1)(\xi-\xi_2)}}, \tag{9}$$

$$B = \sqrt{|\eta^2(1-\vartheta)^2 - (1-\chi)^2|}.$$

Depending on the sign of the expression $\eta^2(1-\vartheta)^2 - (1-\chi)^2$, the constant a_0 takes the values ± 1. The integral I in (9) is an integral of elliptic type dependent on the critical load parameter η, the parameter ϑ characterizing the kind of loading, and the curvature parameter χ. Shells with curvature $\chi < 0$ in whose middle surface the compressive forces $\vartheta > 0$ act, are considered below. This integral reduces to a Legendre normal form [6]. In the general case it can be written in the form

$$I = \mu[A_0 K(k) + A_1 E(k) + A_2 \Pi(h, k)]. \tag{10}$$

Here $K(k)$, $E(k)$, and $\Pi(h, k)$ are, respectively, the complete elliptic integrals of the first, second, and third kinds, $\mu = 2/[\xi_2(1-\xi_1)]^{1/2}$, and the coefficients A_0, A_1, A_2 depend on η, ξ_1, ξ_2, ϑ, χ. In reducing the integral to the form (10) we consider the relationships $|\chi| \leqslant \vartheta$, $|\chi| \geqslant \vartheta$. In the first case the expressions for the coefficients A_0, A_1, A_2 are the following

$$A_0 = 1 - \frac{(1-\vartheta)^2\xi_1^2}{2(1-h)},\ A_1 = -(1-\vartheta)^2\xi_1^2 \frac{h}{2(1-h)(k^2-h)},$$
$$A_2 = -2(1-\vartheta)\xi_1\left[1 - \frac{(1-\vartheta)\xi_1}{2(1-h)} \frac{h^2 - 2h(1+k) + 3k^2}{k^2-h}\right],\ 0 < \eta \leqslant \frac{|\chi|}{\vartheta}, \tag{11}$$

and the constants k^2 and h have the form

$$k^2 = (\xi_2 - \xi_1)/(\xi_2(1-\xi_1)),\ h = (\xi_2 - \xi_1)/\xi_2.$$

The A_0, A_1, A_2 are written in the same form as the parameter η changes within the boundaries $|\chi|/\vartheta \leqslant \eta \leqslant 1$, except in this case $k^2 = \xi_2(1-\xi_1)/(\xi_2-\xi_1)$, $h = 1-\xi_1$. As the parameter η grows further, we obtain

$$A_0 = \left[1 - (1-\vartheta)\xi_1\right]^2 + \frac{(1-\vartheta)^2\xi_1}{2(1-h)},\ A_1 = (1-\vartheta)^2\xi_1^2 \frac{h}{2(1-h)(k^2-h)}, \tag{12}$$
$$A_2 = 2(1-\vartheta)\xi_1\left[1 - (1-\vartheta)\xi_1 \frac{h^2 - 2h(1+k^2) + 3k^2}{4(1-h)(k^2-h)}\right],\ \eta \geqslant 1,$$

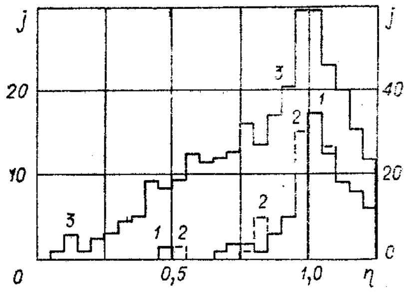

Fig. 1

where

$$k^2 = (\xi_2 - \xi_1)/(\xi_2(1 - \xi_1)),\ h = 1/(1 - \xi_1).$$

Expressions for A_0, A_1, A_2 are obtained analogously under the condition $|\chi| \geqslant \vartheta$. If $0 < \eta \leqslant 1$, then the formulas (11) are valid. These coefficients later take the form

$$A_0 = [1 - (1 - \vartheta)\xi_1]^2 - \frac{(1-\vartheta)}{2(1-h)}[1 - 2\xi_1 - \xi_1^2],\ A_1 = -\frac{(1-\vartheta)^2 h}{2(1-h)(k^2-h)}[1 - 2\xi_1 - \xi_1^2],$$

$$A_2 = -\left\{2(1-\vartheta)[1 - 2\xi_1 + \xi_1\vartheta] - \frac{(1-\vartheta)^2[h^2 - 2h(1+k^2) + 3k^2]}{2(1-h)(k^2-h)}\right.,$$

$$h = \xi_2/(\xi_2 - \xi_1),\ k^2 = (1 - \xi_1)\xi_2/(\xi_2 - \xi_1).$$

And finally, for $\eta \geqslant |\chi|/\vartheta$ the formulas for A_0, A_1, A_2 are the very same as in the relationships (12).

Certain fundamental properties of the density of the spectrum can be extracted in the problems under consideration without evaluating the elliptic integrals. The spectrum of the problems mentioned has two condensation points corresponding to the characteristic eigennumbers $\eta_1 = |\chi|/\vartheta$, $\eta_2 = 1$. For $|\chi| = \vartheta$ these points coincide, resulting in an abrupt growth in the density. The point $\eta = 0$ is asymptotic for the beginning of the spectrum. Analogous results are obtained in vibration problems for shells of negative Gaussian curvature [7].

Results of a numerical experiment on computing the initial section of the spectrum by means of (2) are represented in the figure. The critical loads calculated by means of the formula mentioned are ordered by magnitude, then are grouped in intervals of length $\Delta\eta = 0.05$. The number of eigennumbers per a given interval is denoted by the letter j. On the left is the scale for curves 1 and 2, while on the right is the scale for curve 3. Calculations were performed for the following values of the parameters: curve 1 $\chi = -0.005$, $\vartheta = 0.1$, R/h = 400; curve 2 $\chi = -0.01$, $\vartheta = 0.1$, R/h = 400; and curve 3 $\chi = -0.333$, $\vartheta = 0.5$, R/h = 1600. The condensation point on all three graphs is quite distinct for $\eta = 1$. An increase in the density of the spectrum is observed on curve 3 in the neighborhood of this point. As the thin-walledness parameter increases, the condensation of the eigennumbers in the neighborhood of $\eta = \eta_1$ is more clearly defined. This is related to the fact that estimates of the eigennumber density obtained according to (5) will be more exact the higher the number of the eigennumbers with which we have to deal.

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